Výpočet neurčitého a určitého integrálu

Příklad 1. Vypočtěte metodou per partes:
$\int x^2\cos x dx = (x^2-2)\sin x + 2x\cos x$, $\int (x^2+2x+17)e^x dx = (x^2+17)e^x$, $\int x^2 \operatorname{arctg} x dx = \frac{x^3}{3}\operatorname{arctg} x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1)$, $\int \ln x dx = x(\ln x - 1)$, $\int \sin^2 x dx = \frac{1}{2}(x - \sin x\cos x)$, $\int \frac{\ln x}{x^2}dx = -\frac{1}{x}(\ln x + 1)$, $\int \frac{x}{\sin^2 x}dx = -x\operatorname{cotg} x + \ln|\sin x|$, $\int x\ln x dx = \frac{x^2\ln x}{2} - \frac{x^2}{4}$, $\int e^x\cos x dx = \frac{e^x}{2}(\sin x + \cos x)$, $\int \frac{\ln^2 x}{\sqrt{x}}dx = \sqrt{x}(2\ln^2 x - 8\ln x + 16)$, $\int \arcsin x dx = x\arcsin x + \sqrt{1-x^2}$, $\int \frac{\ln^2 x}{x^2}dx = -\frac{1}{x}(\ln^2 x + 2\ln x + 2)$, $\int x^2\sin(2x)dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x)$, $\int xe^{-x}dx = e^{-x}(-x-1)$, $\int x^3\cos x dx = (x^3-6x)\sin x + (3x^2-6)\cos x$, $\int \operatorname{arctg} x dx = x\operatorname{arctg} x - \frac{1}{2}\ln(x^2+1)$, $\int \cos(\ln x)dx = \frac{x}{2}(\cos(\ln x) + \sin(\ln x))$, $\int \sqrt{x^2+1}dx = \frac{1}{2}(x\sqrt{x^2+1} + \ln|x+\sqrt{x^2+1}|)$.

Příklad 2. Vhodnou substitucí vypočtěte následující triviální integrály:
$\int \frac{\sin^2 x}{\cos^4 x}dx = |\operatorname{tg} x = t| = \frac{1}{3}\operatorname{tg}^3 x$, $\int \frac{1}{x\ln x}dx = |\ln x = t| = \ln|\ln x|$, $\int \frac{\sqrt[3]{\operatorname{arctg} x}}{1+x^2}dx = |\operatorname{arctg} x = t| = \frac{3}{4}\sqrt[3]{(\operatorname{arctg} x)^4}$, $\int \operatorname{arctg}\sqrt{x}dx = |x = t^2| = x\operatorname{arctg}\sqrt{x} - \sqrt{x} + \operatorname{arctg}\sqrt{x}$, $\int \frac{1}{\sqrt{4x+9}}dx = |4x+9 = t| = \frac{1}{2}\sqrt{4x+9}$, $\int \frac{1}{\sin^2(3x-7)}dx = |3x-7 = t| = -\frac{1}{3}\operatorname{cotg}(3x-7)$, $\int \frac{1}{7x-9}dx = |7x-9 = t| = \frac{1}{7}\ln|7x-9|$, $\int \frac{dx}{9+4x^2} = |2x = 3t| = \frac{1}{6}\operatorname{arctg}\frac{2x}{3}$, $\int e^x\cos(e^x)dx = |e^x = t| = \sin(e^x)$, $\int \frac{\sin x}{\sqrt[3]{1+2\cos x}}dx = |1+2\cos x = t| = -\frac{3}{4}\sqrt[3]{(1+2\cos x)^2}$, $\int \frac{e^{\frac{1}{x}}}{x^2}dx = |\frac{1}{x} = t| = -e^{\frac{1}{x}}$, $\int \frac{3^x}{5+3^x}dx = |5+3^x = t| = \frac{1}{\ln 3}\ln|5+3^x|$, $\int \frac{x^3}{\sqrt{1-x^8}}dx = |x^4 = t| = \frac{1}{4}\arcsin x^4$, $\int \frac{1}{x^2}\sin\frac{1}{x}dx = |\frac{1}{x} = t| = \cos\frac{1}{x}$, $\int 2x\sqrt{x^2+1}dx = |x^2+1 = t^2| = \frac{2}{3}(x^2+1)^3$, $\int \frac{x+(\operatorname{arctg} x)^{-1}}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + \ln|\operatorname{arctg} x|$, $\int \frac{dx}{x\cdot\ln x\cdot\ln(\ln x)}dx = \ln|\ln(\ln x)|$, $\int \sin^7 x\cos x dx = |\sin x = t| = \frac{1}{8}\sin^8 x$, $\int \frac{2^x dx}{\sqrt{1+4^x}} = |2^x = t| = \frac{1}{\ln 2}\ln(2^x + \sqrt{1+4^x})$,

Příklad 3. Vypočtěte rychle integrály z funkcí racionálně lomených:
$\int \frac{x^4+6x^2+x-2}{x^4-2x^3}dx = x - 3\ln|x| - \frac{1}{2x^2} + 5\ln|x-2|$, $\int \frac{5}{(2x-3)^3}dx = -\frac{5}{4}\frac{1}{(2x-3)^2}$, $\int \frac{27dx}{\sqrt{2}x-5} = \frac{27}{\sqrt{2}}\ln|\sqrt{2}x-5|$, $\int \frac{8x-31}{x^2-9x+14}dx = 3\ln|x-2| + 5\ln|x-7|$, $\int \frac{xdx}{(x-1)(x+1)^2} = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2(x+1)}$, $\int \frac{11x^2-2x-33}{x^2-3}dx = -\ln|x-\sqrt{3}| - \ln|x+\sqrt{3}| + 11x$, $\int \frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)}dx = \frac{1}{8}\ln|\frac{(2x-1)^3(2x-5)^3}{2x+3}|$, $\int \frac{4-4x}{4x^2-4x+1}dx = -\ln|2x-1| + \frac{1}{2x-1}$, $\int \frac{6x+6}{2x^2+3x}dx = \ln|2x^3+3x^2|$, $\int \frac{3x^4+x^3-5x+2}{x^5-x^4-2x^3}dx = \ln|x(x-2)^3(x+1)^2| - \frac{3}{x} + \frac{2}{x^2}$, $\int \frac{2+2x+x^2-x^3}{2-x^2}dx = \frac{x^2}{2} - x + \sqrt{2}\ln|\frac{\sqrt{2}+x}{\sqrt{2}-x}|$.

Příklad 4. Pečlivě zintegrujte:
$\int \sin^5 x dx = |\cos x = t| = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$, $\int \frac{\cos^5 x}{\sin^4 x}dx = |\sin x = t| = -\frac{1}{3\sin^3 x} + \frac{2}{\sin x} + \sin x$, $\int \frac{dx}{4\sin x - 7\cos x - 7} = |\operatorname{tg}\frac{x}{2} = t| = \frac{1}{4}\ln|4\operatorname{tg}\frac{x}{2} - 7|$, $\int \frac{\sin^3 x}{1+\cos^2 x}dx = |\cos x = t| = \cos x - 2\operatorname{arctg}(\cos x)$, $\int \frac{dx}{\sin x\cos(2x)} = |\cos x = t| = \frac{1}{2}\ln|\frac{1-\cos x}{1+\cos x}| + \frac{1}{\sqrt{2}}\ln|\frac{1+\sqrt{2}\cos x}{-1+\sqrt{2}\cos x}|$, $\int \frac{dx}{sin^4 x} = |\operatorname{tg} x = t| = -\frac{1}{\operatorname{tg} x} - \frac{1}{3\operatorname{tg}^3 x}$, $\int \frac{\sin 2x}{1+\sin^4 x}dx = |\sin x = t| = \operatorname{arctg}(\sin^2 x)$.

Příklad 5. Zintegrujte následující iracionální funkce:
$\int \frac{\sqrt{x}}{1+\sqrt[4]{x^3}}dx = |x = t^4| = \frac{4}{3}(x^{\frac{3}{4}} - \ln|x^{\frac{3}{4}}+1|)$, $\int \frac{1+x}{1+\sqrt{x}}dx = |x = t^2| = 2(\frac{\sqrt{x^3}}{3} - \frac{x}{2} + 2\sqrt{x} - 2\ln|\sqrt{x+1}|)$, $\int \frac{dx}{x\sqrt{2x+1}} = |2x+1 = t^2| = \ln|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}|$, $\int x^2\sqrt[3]{1-x}dx = |1-x = t^3| = -\frac{3}{10}\sqrt[3]{(1-x)^{10}} + \frac{6}{7}\sqrt[3]{(1-x)^7} - \frac{3}{4}\sqrt[3]{(1-x)^4}$, $\int \frac{2+\sqrt{x}}{x(\sqrt[4]{x}+\sqrt[6]{x})}dx = |x = t^{12}| = 4\sqrt[4]{x} - 6\sqrt[6]{x} +$

$12\sqrt[12]{x} + 24\ln|\sqrt[12]{x}| - \frac{24}{\sqrt[12]{x}} - \frac{12}{\sqrt[6]{x}} - 36\ln|\sqrt[12]{x}+1|$, $\int \frac{1-\sqrt[3]{x}}{\sqrt{x}}dx = |x = t^6| = 2\sqrt{x} - \frac{6}{5}x^{\frac{6}{5}}$, $\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}} = |x = t^4| = 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln|\sqrt[4]{x}+1|$, $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} = |x = \sin t| = \frac{x}{\sqrt{1-x^2}}$, $\int \frac{\sqrt{x^2-4}}{x}dx = |x = \frac{2}{\sin t}| = \sqrt{x^2-4} + 2\arcsin\frac{2}{x}$.

Příklad 6. Vypočtěte následující určité integrály:
$\int_0^1 x\mathrm{arctg}\, x dx = |\text{per partes}| = \frac{\pi}{4} - \frac{1}{2}$, $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{(1-\sin^2 x)\cos x}{\sin^2 x}dx = |\sin x = t| = \frac{1}{2}$, $\int_1^2 \frac{xdx}{(x^2+1)^{\frac{3}{2}}} = |x^2+1 = t^2| = -\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{2}}$, $\int_1^9 \frac{dx}{1+\sqrt{x}} = |x = t^2| = 4 - 2\ln 2$, $\int_1^4 \frac{dx}{(1+x)\sqrt{x}} = |x = t^2| = 2\mathrm{arctg}\, 2 - \frac{\pi}{2}$, $\int_4^5 \frac{\sqrt{x-4}}{1+\sqrt{x-4}} = |x - 4 = t^2| = -1 + 2\ln 2$, $\int_0^{\frac{\pi}{4}} \sin x \cos^2 x dx = |\cos x = t| = \frac{1}{12}(4-\sqrt{2})$, $\int_3^8 \frac{dx}{(2+x)\sqrt{1+x}} = |1 + x = t^2| = 2\mathrm{arctg}\, 3 - 2\mathrm{arctg}\, 2$, $\int_2^5 \frac{x-1}{\sqrt{4x-2}}dx = |4x - 2 = t^2| = \frac{3\sqrt{2}}{2}$, $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos^2 x+1}dx = |\cos x = t| = -1 + \frac{\pi}{2}$, $\int_{\ln 2}^{\ln 3} \frac{e^x dx}{e^{2x}-1} = |e^x = t| = \frac{1}{2}(\ln\frac{1}{2} - \ln\frac{1}{3})$, $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x dx = |\cos x = t| = \frac{1}{3} - \frac{1}{5}$.

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